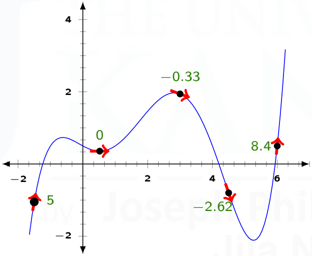
Section 14.3
Partial Derivatives

▸ Prelecture Review Video

Derivatives & Rates of Change - Calculus of 1-Var

For a single-variable function $f(x)$ the **derivative** $f'(a)$ at the point $(a, f(a))$ is the rate of change of $f(x)$ at $x = a$.

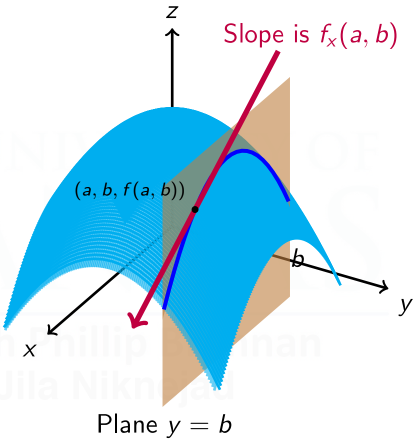


For a function of one variable, we are only concerned with the rate of change as the input variable changes in **one direction** (left/right), but for functions of two or more variables, there are **infinitely many directions**.

1 Geometry of Partial Derivatives

Partial Derivatives

- Let $P = (a, b)$ be a point in the domain of a function $z = f(x, y)$.
- Finding the cross-sectional by fixing $y = b$ results in a single-variable function $g(x) = f(x, b)$; the graph of $g(x)$ lies on the graph of $f(x, y)$.
- Since g is a function of x, we can calculate the derivative $g'(x)$.
- $g'(a)$ is the slope of the tangent line to the graph of $g(x)$ at $x = a$.



The derivative $g'(a)$ is called the **partial derivative** of $f(x, y)$ with respect to the x-variable at the point a (a, b).

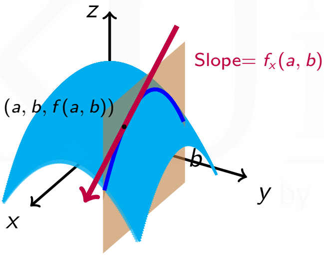
Link

2 The Definition of Derivative and Partial Derivatives

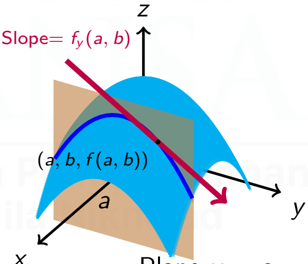
Notations and Definition of Partial Derivatives

The **partial derivative** of $f(x, y)$ with respect to x at (a, b) is **denoted**

$$\underbrace{\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = \frac{\partial f}{\partial x}(a, b) = f_x(a, b)}_{\text{Notations}} = \underbrace{\lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h}}_{\text{Definition}}.$$



Plane $y = b$



Plane $x = a$

The **partial derivative** of $f(x, y)$ with respect to y at (a, b) is **denoted**

$$\left.\frac{\partial z}{\partial y}\right|_{(a,b)} = \frac{\partial f}{\partial y}(a, b) = f_y(a, b) = \lim_{h\to 0} \frac{f(a, b+h) - f(a, b)}{h}.$$

3 Computing Partial Derivatives

Partial Derivatives

To compute the partial derivative $f_x(a,b)$, apply the usual rules for differentiation, treating x as a variable and y as a constant.

To compute the partial derivative $f_y(a,b)$, apply the usual rules for differentiation, treating y as a variable and x as a constant.

Warning: Partial differentiation is **<u>not</u>** implicit differentiation!

Example 1: Calculate the partial derivatives of $f(x,y) = xy - y^2$ at $(1,2)$.

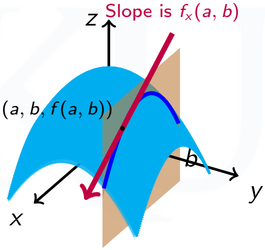
<u>Solution:</u> $\quad f_x(x,y) = y \qquad f_y(x,y) = x - 2y$

- Near $(x,y) = (1,2)$, if we hold y fixed at 2 and let x vary, then the instantaneous rate of change of z is $\frac{\partial f}{\partial x}(1,2) = 2$.
- Near $(x,y) = (1,2)$, if we hold x fixed at 1 and let y vary, then the instantaneous rate of change of z is $\frac{\partial f}{\partial y}(1,2) = -3$.

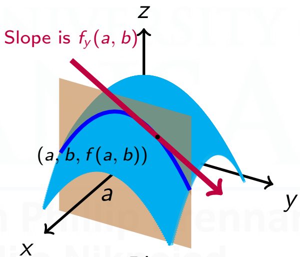
4 Tangent Lines on Surfaces in x or y Directions

Geometry of Partial Derivatives

The planes $x = a$ and $y = b$ intersect the surface $z = f(x, y)$ in curves $z = f(a, y)$ and $z = f(x, b)$ (respectively). The partial derivatives are the slopes of the tangent lines to the two curves.



Plane $y = b$



Plane $x = a$

- The tangent line to the graph of $z = f(x, b)$ contains the point $(a, b, f(a, b))$ and has direction vector $\langle 1, 0, f_x(a, b)\rangle$.
- The tangent line to the graph of $z = f(a, y)$ contains the point $(a, b, f(a, b))$ and has direction vector $\langle 0, 1, f_y(a, b)\rangle$.[1]

[1] An example is in this Video.

Partial Derivatives Examples

Additional Example: Find the partial derivatives of $f(x,y) = x(y+1) - x^2$. Then calculate the tangent lines to the graph of f at $(-2, 2, -10)$ in both the x- and y-directions.

<u>Solution:</u> $f_x(x,y) = y + 1 - 2x$ $\qquad$ $f_y(x,y) = x$

(x) The tangent line to the intersection curve of $f(x,y)$ and the plane $y = 2$ at $\big(-2, 2, f(-2,2)\big)$ is

$$\vec{r_x}(t) = \langle -2, 2, -10 \rangle + t\,\langle 1, 0, 7 \rangle = \langle -2 + t, 2, -10 + 7t \rangle$$

(y) The tangent line to the intersection curve of $f(x,y)$ and the plane $x = -2$ at $\big(-2, 2, f(2,-2)\big)$ is

$$\vec{r_y}(t) = \langle -2, 2, -10 \rangle + t\,\langle 0, 1, -2 \rangle = \langle -2, 2 + t, -10 - 2t \rangle$$

This example is not included in the lecture. ▸ Watch a Video.

5 Partial Derivatives of Functions of Many Variables

Partial Derivatives of Functions of Many Variables

Partial derivatives of functions of n variables are defined in the same way.

For example, in a three-variable function $f(x, y, z)$, we calculate f_z by differentiating with respect to z, treating x and y as constants.

Example 2: Find the partial derivatives of

$$f(x, y, z) = z\ln(x^2 + y^2) + \sin(xz).$$

Answer:

$$f_x(x, y, z) = \frac{\partial f}{\partial x}(x, y, z) = \frac{2xz}{x^2 + y^2} + z\cos(xz)$$

$$f_y(x, y, z) = \frac{\partial f}{\partial y}(x, y, z) = \frac{2yz}{x^2 + y^2}$$

$$f_z(x, y, z) = \frac{\partial f}{\partial z}(x, y, z) = \ln(x^2 + y^2) + x\cos(xz)$$

Video

Higher-Order Partial Derivatives

For $z = f(x, y)$, the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ are again functions of (x, y). The partial derivatives of *these* functions are the **second order partial derivatives** of $f(x, y)$:

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(x, y)\right) = \frac{\partial^2}{\partial x^2} f(x, y) = f_{xx}(x, y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x, y)\right) = \frac{\partial^2}{\partial x\,\partial y} f(x, y) = f_{yx}(x, y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x, y)\right) = \frac{\partial^2}{\partial y\,\partial x} f(x, y) = f_{xy}(x, y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f(x, y)\right) = \frac{\partial^2}{\partial y^2} f(x, y) = f_{yy}(x, y)$$

The second-order partials are useful in classifying critical points (just as in Calculus I), but in a more subtle way. Stay tuned.

Higher-Order Partial Derivatives

Partial derivatives like f_{xx} and f_{zzz} that involve only one variable are called **pure**; partial derivatives like f_{xy} and f_{xyz} that involve more than one variable are called **mixed**.

In general,

$$\frac{\partial^n f}{\partial x_n \dots \partial x_2 \partial x_1} = f_{x_1 x_2 \dots x_n}.$$

For example,

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right) = \frac{\partial^2}{\partial x\,\partial y} f(x,y) = \frac{\partial}{\partial x} f_y(x,y) = f_{yx}(x,y)$$

<u>Good news:</u> Often, the order of partial differentiation does not matter.

Higher-Order Partial Derivatives

Example 3: Compute the first and the second-order partial derivatives of

$$f(x, y) = xe^{xy} + y^2 - y(x^2 + 1).$$

Solution:

$$f_x = \underbrace{e^{xy} + xye^{xy}}_{\text{Product Rule}} - 2xy \qquad f_y = x^2e^{xy} + 2y - (x^2 + 1)$$

$$f_{xx} = 2ye^{xy} + xy^2e^{xy} - 2y \qquad f_{yx} = 2xe^{xy} + x^2ye^{xy} - 2x$$

$$f_{xy} = 2xe^{xy} + x^2ye^{xy} - 2x \qquad f_{yy} = x^3e^{xy} + 2$$

Observation: $f_{xy} = f_{yx}$.

This is not an accident! ▸ Video

6 Clairaut's Theorem

Clairaut's Theorem

Clairaut's Theorem

If $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are continuous, then $f_{xy} = f_{yx}$.

Clairaut's Theorem — General Case

If all k^{th}-order partial derivatives of $f(x_1, \ldots, x_n)$ are continuous, then the order of differentiation does not matter.

This hypothesis holds for all elementary functions (but be careful with functions that are defined piecewise!)

For example, if all third-order partial derivatives of $f(x, y, z)$ are continuous, then

$$f_{xyz} = f_{xzy} = f_{yxz} = f_{yzx} = f_{zxy} = f_{zyx}.$$